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BERNOULLIS ARE STANDARD WHEN ENTROPY IS NOT AN OBSTRUCTION

BY

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ABSTRACT

In this paper we look at r equivalence, which is an equivalence relation that is implicit in Vershik's classification of r-adic decreasing sequences of σ-algebras, and also in work of Stepin. This equivalence relation is used to classify group actions of the group $G = \sum_{n=1}^{\infty} \mathbf{Z}/r_n\mathbf{Z}$, $r = (r_1, r_2, ...)$. It is shown that given any sequence of natural numbers satisfying a certain growth rate, all Bernoulli G actions are r equivalent to a certain natural action of G, which we call the translation action. Furthermore, these actions are zero r entropy and r finitely determined, where this notion arises canonically from the restricted orbit equivalence theory of Kammeyer and Rudolph.

1. Introduction

Let $\mathbf{Z}$ be the integers, and $\mathbf{N}$ be the natural numbers. Let $(X, \mathcal{F}, \mu)$ be a non-atomic Lebesgue probability space, and $\{T_g\}_{g\in G}$ be a free measure preserving ergodic group action on the space, where $G = \sum_{n=1}^{\infty} \mathbf{Z}/r_n\mathbf{Z}$ and $r_n \in \mathbf{N} \setminus \{1\}$. Set $r = (r_1, r_2, \ldots)$. Let $G_i = \sum_{n=1}^{i} \mathbf{Z}/r_n\mathbf{Z}$. Set $q_i = \#G_i = \prod_{n=1}^{i} r_n$. Notice $G = \bigcup G_i$. Our first example of a G action is what we will call the **translation** action.

Example 1.1: Let $X = \prod_{n=1}^{\infty} \mathbf{Z}/r_n\mathbf{Z}$, $\mu =$ Haar measure, $\mathcal{F} =$ the completion of the Borel σ-algebra and $(T_g x)_n = (x_n + g_n) \bmod r_n$. Since the action is rotation on a compact abelian group it has discrete spectrum and entropy 0.

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Another example is the natural generalization of the Bernoulli actions for this group.

Example 1.2: Let $X = \{0, \dots, n-1\}^G$, μ be the Bernoulli measure on X given by a probability vector $(p_0, \dots, p_{n-1})$ on the factors, let $\mathcal{F}$ be the completion of the Borel σ-algebra, and T be the shift. Notice that the partition P_0 by the values of the identity coordinate is a generator for this action, so

$$h(T) = h(T, P) = -\sum_{i=0}^{n-1} p_i \log p_i.$$

There is a natural equivalence relation that arises when considering group actions of this group, namely r equivalence.

Definition 1.1: Suppose T and S are two $\sum \mathbf{Z}/r_n\mathbf{Z}$ actions. T is r equivalent to S if there is a 1-1 measure preserving map $\phi : X \to Y$ such that $\{\phi(T_g x)\}_{g \in G_i} = \{T_g \phi x\}_{g \in G_i}$ for every i.

Dye's theorem says that any two ergodic G actions are orbit equivalent, since G is a countable discrete amenable group. However, this equivalence relation asks for more than just orbit equivalence, since it asks to preserve each G_i orbit. This equivalence relation arises naturally from classifying certain decreasing sequences of σ-algebras up to isomorphism. Vershik initiated the study of these systems in the late 1960's. A sequence of non-atomic σ-algebras $\mathcal{F} = \mathcal{F}_0 \supset \mathcal{F}_1 \cdots$ is r-adic if the fibers of $\mathcal{F}_n$ over $\mathcal{F}_{n+1}$ all contain r_n points of equal mass. Two of these sequences are isomorphic if there is a 1-1 measure preserving map between the two spaces that takes the nth σ algebras to each other. Choosing measurable sections of $\mathcal{F}_n | \mathcal{F}_{n+1}$ gives an action of G. Likewise, given any G action it is possible to define such a sequence by looking at the σ-algebras of G_n invariant sets. It follows from the definitions that isomorphism between r-adic σ-algebras is the same as the notion of r equivalence between two G actions. A G action is ergodic iff its corresponding sequence has trivial intersection. An r-adic sequence $\{\mathcal{F}_n\}$ is called **standard** if there exists a sequence of independent partitions $\{P_i\}$ such that $\mathcal{F}_n = \bigvee_{i=n}^{\infty} P_i$. This is Vershik's terminology. The sequence of σ-algebras generated by the G_i invariant sets of the translation action is standard. Vershik originally thought that every r-adic sequence with trivial intersection was standard [14]. This would imply that any two ergodic G actions are r equivalent. Later, Vershik constructed examples of dyadic sequences that are not standard [15]. Later, Vershik showed that a certain entropy which he defined for r-adic sequences is an isomorphism invariant in the case when the sequence r does not grow too fast. He also showed that this entropy corresponds to the regular group entropy in the case of Bernoulli actions [16] [17]. Stepin showed that the entropy that Vershik defined corresponds to the regular group action entropy for arbitrary G action, and reproved the invariance of entropy when r does not grow too fast [12] [13]. In particular, this condition encompasses dyadic equivalence, which is the case when $r_n = 2\ \forall n$. Vershik also showed, through his lacunary isomorphism theorem, that there exist sequences r for which entropy is not an r equivalence invariant [14]. Thus a natural question arises: When is entropy an invariant for r equivalence? The previously mentioned theorem gives a partial solution to this. More precisely, define

$$h_r(T) = \inf\{h(S) \mid S \text{ is } r \text{ equivalent to } T\}.$$

THEOREM 1.1 ([18]): *If* $\sum \frac{\log r_{k+1}}{q_k} < \infty$ *then* $h_r(T) = h(T) \quad \forall T$.

More recently, Hasfura showed that there is a mixing G action in every r equivalence class [6]. This implies that for any r, r equivalence is strictly weaker than an isomorphism of G actions. This is because mixing is a G action isomorphism invariant and the translation action is not mixing. Building on this result, Hasfura and Fieldsteel showed that there is a completely positive entropy G action in every positive entropy equivalence class [5]. Also, we showed that r equivalence is a restricted orbit equivalence [7] as defined in Kammeyer and Rudolph's recent paper [8]. In other words, for any r there is a corresponding 'size' m_r, whose equivalence classes correspond to those of r equivalence. Using this fact and some combinatorial arguments, one gets the following.

THEOREM 1.2 ([7]): *If* $\sum \frac{\log r_{k+1}}{q_k} = \infty$ *then* $h_r(T) = 0 \quad \forall T$.

In the first case, r is called **entropy preserving** and in the second case r is called **entropy free**. Restricted orbit equivalence theory ensures that the r entropy is achieved on a residual set in every r equivalence class. The goal of this paper is to determine the nature of the standard equivalence class. In the next two sections, a number of different properties of an action are defined, and then proven to be equivalent to standardness. One of the properties defined in section 2 is the same as Vershik's standardness criterion, which he proves is equivalent to standardness [18]. The property defined in section 3 is r finitely determined. The definition of r finitely determined follows a similar pattern to the definition of finitely determined for Ornstein's isomorphism theorem [10] and of finitely fixed for the theory of Kakutani equivalence, which was developed by Feldman, Katok, Ornstein, Weiss and Rudolph [2], [9], [11]. All the directions of the implications are included for completeness, although the equivalence of the standardness criteria to standardness was originally proved directly and hence does not depend on restricted orbit equivalence theory. The main theorem in the last section states that when r is entropy free, the Bernoullis are standard. This is proved using one of the standardness criteria. Using this result, together with a certain extension of Sinai's theorem, Feldman proved that there is a positive entropy action in every r equivalence class when r is entropy free [3]. Then using this extension, Feldman proved that there are actions of any entropy in every equivalence class [4]. For simplicity, when we want to denote the G action we drop T and simply write gx.

2. Standardness criterion

Set $G = \sum \mathbf{Z}/r_n\mathbf{Z}$, $r = (r_1, r_2, \ldots)$, and $G_i = \sum_{n=1}^{i} \mathbf{Z}/r_n\mathbf{Z}$. In this section and the next we define a number of different properties for G actions and show that they are all equivalent. Define

$$\mathcal{A}_n = \{a \in G_n^{G_n} : a(g) - a(h) \in G_i \Leftrightarrow g - h \in G_i, \quad \forall i \leq n, \quad \forall g, h \in G_n\}.$$

These are automorphisms of the G_n tree. For a finite partition P, let

$$\alpha_n^P(x,y) = \inf_{a \in \mathcal{A}_n} \frac{\#\{g \in G_n | P(gx) \neq P(a(g)y)\}}{q_n}.$$

Criterion 1 states that for all finite partitions P,

$$\int \alpha_n^P(x,y) d\mu \times \mu \to 0.$$

This is the criterion given in [18]. If $a \in \mathcal{A}_m^0$ for $m \geq n$ define

$$m_n(a) = \sum_{j=1}^{n} \frac{\#\{g \in G_j | a(g) - a(0) \neq g\}}{2^j q_j}.$$

$m_n(a)$ is small if on most of the small subgroups a looks like a translation by some element $g \in G_n$. Let

$$v_n^P(x,y) = \inf_{a \in \mathcal{A}_n} \left[\frac{\#\{g \in G_n | P(gx) \neq P(a(g)y)\}}{G_n} + m_n(a) \right].$$

Criterion 2 states that for a generating partition P,

$$\int v_n^P(x,y) d\mu \times \mu \to 0.$$

The goal of the next two sections is to prove the following.

THEOREM 2.1: *Given a G action T with a finite generator, TFAE:*

1. *T is standard,*
2. *T satisfies criterion 1,*
3. *T satisfies criterion 2,*
4. *T is r finitely determined and $h_r(T) = 0$,*

where the last property has not yet been introduced. Before doing so, however, we show $1 \Rightarrow 2 \Rightarrow 3$. After that, we define 4 and show $3 \Rightarrow 4 \Rightarrow 1$.

Remark 2.1: When T does not have a finite generator, simple modifications can be made in the proof of this theorem to establish $1 \Rightarrow 2 \Rightarrow 4 \Rightarrow 1$.

To show $1 \Rightarrow 2$ it suffices to show that the translation action satisfies criterion 1 and that criterion 1 is an r equivalence invariant.

PROPOSITION 2.1: *The translation action satisfies criterion 1.*

Proof: First notice that the translation action has the property that there is a sequence of sets $\{F_n\}$ such that $\forall n \quad \bigcup_{g\in G_n} gF_n = X, \quad \{gF_n\}_{g\in G_n}$ are mutually disjoint, and if $\mathcal{F}_n = \{gF_n\}_{g\in G_n}$, then $\mathcal{F}_n \nearrow \mathcal{F}$. This is because the sets $F_n = \{x \in X| \quad x_1 \cdots x_n = 0 \cdots 0\}$ serve as the bases of towers $\mathcal{F}_n$ that increase to the full algebra. Hence it suffices to show that an action with this property satisfies criterion 1. Set $L_n(P) = \int \alpha_n^P(x,y) d\mu \times \mu$ and $L(P) = \lim_{n\to\infty} L_n(P)$. This limit exists since L_n is decreasing in n. Since for any $\mathcal{F}_j$ there is only one G_j name up to translation by elements in G_j, $L(\mathcal{F}_j) = 0$. Also L is continuous. Since any finite partition P can be approximated arbitrarily well by some $\mathcal{F}_j$ and, for all j, $L(\mathcal{F}_j) = 0$, it follows that for any finite partition P, $L(P) = 0$. ∎

Now we show that criterion 1 is an r equivalence invariant. Suppose T is a G action on $(X, \mathcal{F}, \mu)$ with finite partition P, S is the standard action on $(Y, \mathcal{G}, \nu)$ and $\phi : X \to Y$ is an r equivalence. Set $\phi P(x) = P(\phi^{-1}x)$. For $\phi x, \phi y \in Y$, find $a \in \mathcal{A}_n$ that attains the minimum in $\alpha_n^{\phi P}(\phi x, \phi y)$. For $g \in G_n$, let h be defined by

$$\phi^{-1} S_g \phi y = T_h y.$$

Define $b(h)$ by

$$\phi^{-1} S_{a(g)} \phi x = T_{b(h)} x.$$

From these formulas it is clear that

$$\alpha_n^P(x,y) = \alpha_n^{\phi P}(\phi x, \phi y)$$

and the result follows.

Now we prove 2 $\Rightarrow$ 3. For a generator P and any j it is possible to find n such that for most pairs of points (x,y), $\alpha_n^{P^j}(x,y)$ is small. This implies that the automorphism used in $\alpha_n^{P^j}(x,y)$ could have been chosen as a translation on the G_j word of many of the points in the G_n orbit of (x,y), which means that v_n^P is also small.

3. r finitely determined

In this section we prove criterion 2 implies zero r entropy and r finitely determined implies standard. Restricted orbit equivalence theory does not guarantee the existence of a 'finitely determined' class for an entropy free equivalence relation. However, the results of this section show that they exist, and that they correspond to the actions in the standard equivalence class. We first define r finitely determined in two ways. We develop the idea of $\bar{v}^r$, which is the analogue of $\bar{d}$ in this setting. We develop a finite version, $\bar{v}_n^r$, and then establish the connection between the two. By using the finite version, we show that criterion 2 implies r finitely determined. To close the circle of implications, we then apply the equivalence theorem, which implies that zero r entropy and r finitely determined imply standard.

Let $(X,\mathcal{F},\mu,P)$ and $(Y,\mathcal{G},\nu,Q)$ be two free ergodic measure preserving G actions. For any partition P define $P^n = \bigvee_{g\in G_n} gP$, $\mu_n = (P^n)^*\mu$, and likewise for Q.

Let $\mathcal{A}_n^0 = \{a \in \mathcal{A}_n | a(0) = 0\}$. $G_n \times G_n$ acts on $P^n \times Q^n$ by the shift and G_n acts on $\mathcal{A}_n^0$ by $ga(k) = a(g+k) - a(g)$. Let $\hat{X}_n = P^n \times Q^n \times \mathcal{A}_n^0$ and $\hat{T}^n$ be the corresponding action on $\hat{X}_n$. Let $\pi_1 : \hat{X}_n \to P^n$ and $\pi_2 : \hat{X}_n \to Q^n$ be projection and $\phi_{m,n} : \hat{X}_m \to \hat{X}_n$ be the restriction map. Let $aq(g) = q(a^{-1}(g))$. By an abuse of notation, μ can be a measure on X or P^G, and likewise ν can be a measure on Y or Q^G. Let $A(\hat{p},\hat{q},a) = (a\hat{p}, a\hat{q}, a^{-1})$. Define

$$J_n^r(\hat{X}_n) = \{\hat{\mu} \in \mathcal{M}(\hat{X}_n) | \hat{T}^n\hat{\mu} = \hat{\mu},\ \pi_1^*\hat{\mu} = \mu_n, (\pi_2 A)^*\hat{\mu} = \nu_n\}.$$

Remember, if $a \in \mathcal{A}_m^0$ for $m \geq n$,

$$m_n(a) = \sum_{j=1}^{n} \frac{\#\{g \in G_j | a(g) \neq g\}}{2^j q_j}.$$

Define

$$\bar{v}_n^r(\mu_n,\nu_n) = \inf_{\hat{\mu}\in J_n^r} \left(\hat{\mu}\{\hat{p}(0) \neq \hat{q}(0)\} + \int m_n(a)d\hat{\mu}\right).$$

PROPOSITION 3.1: $\bar{v}_n^r(\mu_n, \nu_n) \geq \bar{v}_{n-1}^r(\mu_{n-1}, \nu_{n-1})$.

Proof: For any $\hat{\mu}_n \in J_n^r$ define the projection to be $\phi_{n,n-1}^* \hat{\mu}_n$. This new measure is in J_{n-1}^r since it projects correctly and is invariant under $\hat{T}^{n-1}$. Since this projection measure does not increase the value in the expressions in the definition of $\bar{v}_n^r$, the result follows. ∎

Given $\hat{\mu}_n \in J_n^r(\hat{X}_n)$ it is possible to construct $\hat{\mu}_{n,\infty}$ on $\hat{X} = (P \times Q)^G \times \mathcal{A}$ where

$$\mathcal{A} = \{a \in G^G | a(0) = 0, a(h) - a(g) \in G_j \Leftrightarrow h - g \in G_j \forall j, \forall g, h \in G\}.$$

If $a \in \mathcal{A}$ then $a(G_j) = G_j \forall j$. This forces $\mathcal{A}$ to be compact, since a diagonalization argument shows that any sequence has a convergent subsequence. Now $\hat{\mu}_{n,\infty}$ will only give support to those $a \in \mathcal{A}$ such that $a(0, g) = (0, g)$, where $0 \in G_n$ and $g \in G/G_n$. This implies that $a|_{gG_n} \in \mathcal{A}_n^0$. To define $\hat{\mu}_{n,\infty}$ it is enough to specify for $m \geq n$ that

$$\hat{\mu}_{n,\infty}((\hat{p}, \hat{q}, a)|_{G_m}) = \prod_{g \in G_m / G_n} \hat{\mu}_n((\hat{p}, \hat{q}, a)|_{gG_n}).$$

It is clear that $\phi_{\infty,n}^* \hat{\mu}_{n,\infty} = \hat{\mu}_n$. Notice that $\hat{\mu}_{n,\infty}$ is G invariant, since if $g \in G_n$, $\hat{\mu}_n$ is G_n invariant, and if $g \in G_m/G_n$ then it is invariant because of the multiplication property. However, in general $\hat{\mu}_{n,\infty}$ will not project correctly. Take $\hat{\mu}_\infty$ to be a weak star limit point of $\{\hat{\mu}_{n,\infty}\}$. Notice now that $\hat{\mu}_\infty$ projects correctly. To see this, notice that given any cylinder set based on coordinates in some G_m, $\hat{\mu}_{n,\infty}$ projects correctly for $n \geq m$. For $a \in \mathcal{A}$, let $m(a) = \lim m_n(a)$. This limit exists since m_n is increasing in n. Notice now that

$$\lim \bar{v}_n^r(\mu_n, \nu_n) = \hat{\mu}_\infty\{(p, q, a) \in \hat{X} | \hat{p}(0) \neq \hat{q}(0)\} + \int m(a) d\hat{\mu}_\infty.$$

Set $\bar{v}^r(\mu, \nu) = \lim \bar{v}_n^r(\mu_n, \nu_n)$. Define $|P, Q| = \frac{1}{2} \sum |\mu p_i - \nu q_i|$.

Definition 3.1: A G process $(X, \mathcal{F}, \mu, P)$ is r finitely determined if given any $\epsilon > 0$, $\exists j$ and $\delta > 0$, such that whenever $(Y, \mathcal{G}, \nu, Q)$ is another ergodic G process with

1. $|P^j, Q^j| < \delta$, and
2. $h_r(T, P) < h_r(S, Q) + \delta$, then
3. $\bar{v}^r(\mu, \nu) < \epsilon$, or equivalently,
4. $\bar{v}_n^r(\mu_n, \nu_n) < \epsilon \quad \forall n$.

First we show that if some G process $(X, \mathcal{F}, \mu, P)$ satisfies criterion 2, then $h_r(T) = 0$. Let $\bar{d}$ be the metric between words that counts the number of disagreements and divides by the length of the words. If this process satisfies the standardness criterion, then most of the pairs of words can be matched most of the time in $\bar{d}$ after applying an automorphism of the tree. For each G_n word apply an automorphism to it so that it is within ϵ in $\bar{d}$ to all but ϵ of all the other words in P^n. This new system is r equivalent to the old one but has entropy 0, since given any $\epsilon > 0$ there exists an n such that after throwing away a set of measure ϵ all the remaining G_n words are within ϵ of each other. Now we show that if some G process $(X, \mathcal{F}, \mu, P)$ satisfies criterion 2, then it is r finitely determined.

THEOREM 3.1: *Given $\epsilon > 0$ and $\int v_i^P d\mu \times \mu \to 0$, there is a j and $\delta > 0$ such that whenever $(Y, \mathcal{G}, \nu, Q)$ is another ergodic process with $|P^j, Q^j| < \delta$ then $\exists N$ such that, for $n \geq N$, $\bar{v}_n^r(\mu_n, \nu_n) < \epsilon$.*

Proof: Since the partition is fixed, let $v_j(x,y) = v_j^P(x,y)$. Notice v_j is a function on $X \times X$ or on $P^j \times P^j$. Set $\delta = (\epsilon/32)^2$. Find j such that $\int v_j(p', p) d\mu_j \times \mu_j < \delta$. By assumption $|P^j, Q^j| < \delta$. Let $w \in P^j$ have the property that $\int v_j(w,p) d\mu_j < \delta$. Set

$$\mathcal{P}_j = \{p \in P^j | v_j(w,p) \leq \epsilon/32\}.$$

$\mu_j \mathcal{P}_j > 1 - \epsilon/32$. Since $|P^j, Q^j| < \delta$, there is a map $\pi : P^j \to Q^j$, 1-1 and measure preserving, and a set D such that if $p \in D$ then p and πp have the same G_j name, and $\mu_j D > 1 - \delta$. Now

$$\int v_j(w,q) d\nu_j = \int_{\pi D} v_j(w,q) d\nu_j + \int_{\pi D^c} v_j(w,q) d\nu_j \tag{1}$$

$$\leq \int_D v_j(w,q) d\mu_j + \delta \tag{2}$$

$$\leq 2\delta. \tag{3}$$

Let $\mathcal{Q}_j = \{q \in Q^j | v_j(w,q) \leq \epsilon/16\}$. $\nu \mathcal{Q}_j > 1 - \epsilon/16$. If $p \in \mathcal{P}_j$ and $q \in \mathcal{Q}_j$ then $v_j(p,q) \leq \epsilon/8$. Call a word in P^n **good** if the fraction of G_j coset names in that word that fall in $\mathcal{P}_j$ is greater than $1 - \epsilon/16$, and likewise for Q^n. Define

$$B_n^P = \{p \in P^n | p \text{ is good}\}$$

and

$$B_n^Q = \{q \in Q^n | q \text{ is good}\}.$$

Notice both these sets are G_n invariant. By the ergodic theorem, pick N such that for all $n \geq N$, $\mu B_n^P > 1 - \epsilon/16$, and $\nu B_n^Q > 1 - \epsilon/16$. Fix any $n \geq N$ and build a tower in P^n over a base F_1 of height q_n and likewise in Q^n with base F_2. Let $F = F_1 \times F_2$. For each pair of names, $p_F = p \cap F_1$ and $q_F = q \cap F_2$, with $p \in P^n, q \in Q^n$, there is a $q' = gq_F$ for some $g \in G_n$ such that some a that attains the minimum in $v_n(p_F, q')$ fixes the identity. For any $q_F \in F_2$, choose such an a and q'. The measure will be supported on points $(\hat{p}, \hat{q}, a)$ (and translates of these points), where $\hat{p} = p_F$ and $\hat{q} = aq'$. Let

$$\hat{\mu}(\hat{p}, \hat{q}, a) = \frac{(\mu p_F)(\nu q_F)}{q_n(\mu \times \nu)F} = \hat{\mu}(g(\hat{p}, \hat{q}, a)).$$

Notice that $\hat{\mu}\hat{X} = 1$ and $\hat{\mu}$ is G_n invariant. Furthermore, $\hat{\mu}$ projects correctly. It suffices to check this for the q' specified beforehand:

$$\hat{\mu}\{(\hat{p}, \hat{q}, a) | a\hat{q} = q'\} = \sum_{p_F \in F_1} \hat{\mu}\{(\hat{p}, \hat{q}, a) | a\hat{q} = q', \pi_1 \hat{p} = p_F\} \tag{4}$$

$$= \frac{\nu q_F}{q_n \nu F_2} = \nu q'. \tag{5}$$

Let $\hat{x} = (\hat{p}, \hat{q}, a)$, and $E = B_n^P \times B_n^Q \cap F_1 \times F_2$. Notice that $\mu \times \nu E \geq (1-\epsilon/8)\mu \times \nu F$ and, if $(p, q) \in E$, then $v_n(p, q) \leq \epsilon/4$. It follows that

$$\bar{v}_n^r(\mu_n, \nu_n) \leq \hat{\mu}\{\hat{p}(0) \neq \hat{q}(0)\} + \int m_n(a) d\hat{\mu} \tag{6}$$

$$= \int \left(\frac{\#\{g \in G_n | p(g) \neq q(a(g))\}}{G_n} + m_n(a) \right) d\hat{\mu} \tag{7}$$

$$= \int v_n(p, q) d\hat{\mu} \tag{8}$$

$$= \frac{1}{(\mu \times \nu)F} \int_F v_n(p, q) d\mu \times \nu \tag{9}$$

$$\leq \frac{1}{(\mu \times \nu)F} \int_E v_n(p, q) d\mu \times \nu + \frac{\epsilon}{8} < \epsilon. \tag{10}$$

This finishes $3 \Rightarrow 4$.

Remark 3.1: Given any restricted orbit equivalence with size m, Kammeyer and Rudolph define a canonical notion of m finitely determined [8]. Since r equivalence is a restricted orbit equivalence with corresponding size m_r [7], there is a corresponding notion of m_r finitely determined. This definition coincides with the definition of r finitely determined.

For $4 \Rightarrow 1$, apply the equivalence theorem proved in [8], which states that any two m_r finitely determined actions of equal r entropy are m_r equivalent. This translates as follows: Any two r finitely determined actions of zero r entropy are

r equivalent. Since the translation action is r finitely determined with zero r entropy, if T is r finitely determined with zero r entropy, then T is standard.

4. Bernoullis are standard

Let $G = \sum_{n=1}^{\infty} \mathbf{Z}/r_n\mathbf{Z}$, $r = (r_1, r_2, \ldots)$, $G_i = \sum_{n=1}^{i} \mathbf{Z}/r_n\mathbf{Z}$, and $q_i = \#G_i$. Let $X = (\{0, \ldots, l-1\}^G, \mathcal{F}, \mu, \sigma)$ where μ is iid measure $(1/l, \ldots, 1/l)$. Let P be the partition at the identity into l sets, and $P^i = \bigvee_{g \in G_i} \sigma_g P$. Let $r_{i+1} = l^{a_{i+1} q_i}$. The goal of this section is to prove the following.

THEOREM 4.1: *If $\sum a_k = \infty$ then X is standard.*

The main tool used in this proof is the standardness criterion developed in the previous section, i.e. given $\epsilon > 0$ there exists j such that

$$\int v_j^P(x,y) d\mu \times \mu < \epsilon.$$

The idea for the proof is as follows. Given any two G_j words, apply an inductive algorithm to match the words in the v_j metric. Step 1 switches G_{j-1} cosets around in the G_j word so that most of the G_{j-1} coset pairs have the same name on the first $\tilde{a}_j q_{j-1}$ coordinates. In each of these steps, $\tilde{a}_k$ is less than half of a_k, and will be specified later. Now the first $\tilde{a}_j q_{j-1}$ coordinates of every G_{j-1} coset pair (even the unmatched ones) will never be tampered with again. At step 2, switch around the remaining G_{j-2} cosets inside every G_{j-1} word so that the name of the first $\tilde{a}_{j-1} q_{j-2}$ coordinates in most of the G_{j-2} coset pairs agree. Repeat this process until the $(j-i)$th stage. The inside of each G_i coset pair remains the same, so that if i is chosen large enough, the size of the automorphism that was used is small. Choose j large enough so that after the algorithm is applied $j - i$ times, most pairs of words will agree most of the time.

The following series of remarks ensures that it is possible to pick $\tilde{a}_{k+1}$ so that $\tilde{a}_{k+1} r_k \in \mathbf{N}$ and $\sum \tilde{a}_k = \infty$. In particular this will imply that the number of coordinates reserved at each stage, namely $\tilde{a}_{k+1} q_k$, is a whole number of G_{k-1} cosets.

Set $B = \{k : a_k \leq 1/k^2\}$. Then $\sum_{B^c} a_k = \infty$, since $\sum_B a_k < \infty$. WLOG assume that $a_k > 1/k^2$ for every k, since skipping steps in the matching algorithm gives no change in the estimates. With this reduction, $\sum 1/r_k < \infty$ since

$$\frac{1}{r_k} = \frac{1}{l^{a_{k+1} q_k}} \leq \frac{1}{(l)^{2^k / 2k^2}}$$

which is summable. Now

$$\sum \left[\frac{\log r_{k+1}}{q_k} \right] \frac{1}{r_k} \geq \sum \left(\frac{\log r_{k+1}}{q_k} - \frac{1}{r_k} \right) = \infty.$$

Set

$$\tilde{a}_{k+1} = \min\left\{\left\lceil\frac{\log r_{k+1}}{q_k}\right\rceil\frac{1}{2r_k}, 1-\epsilon\right\}.$$

From the previous remarks, it follows that $\sum \tilde{a}_k = \infty$.

The following lemma gives an estimate on the amount of matching gained in one step of the algorithm. Let d_k be the probability that a pair of G_k words agree on the first $\tilde{a}_{k+1}q_k$ coordinates. Since the coordinates are independent and each symbol has equal weight,

$$d_k = l^{-\tilde{a}_{k+1}q_k}.$$

A G_{k+1} pair of words is **good** if the last $(1-\tilde{a}_{k+2})r_{k+1}$ G_k cosets can be rearranged so that $1-\epsilon/8$ of those G_k coset pairs agree on the first $\tilde{a}_{k+1}q_k$ coordinates. Let B_k be the set of all good G_k pairs and $w_k = \mu \times \mu B_k$. We want to estimate w_{k+1}. For a pair of G_{k+1} words (p,q), look at the first G_k coset of p. The probability of finding a G_k coset of q to match it is

$$d_k + d_k(1-d_k) + \cdots + d_k(1-d_k)^{r_{k+1}(1-\tilde{a}_{k+2})-1} = (1-(1-d_k)^{r_{k+1}(1-\tilde{a}_{k+2})-1}).$$

Likewise, the probability of matching the nth G_k coset is

$$(1-(1-d_k)^{r_{k+1}(1-\tilde{a}_{k+2})-n}).$$

So the probability of matching $N_k = [(1-\epsilon/8)(1-\tilde{a}_{k+2})r_{k+1}]+1$ of the cosets can be estimated by

$$m_{k+1} \geq \prod_{n=1}^{N_k}(1-(1-d_k))^{r_{k+1}(1-\tilde{a}_{k+2})-n}).$$

The following lemma, which demonstrates that m_{k+1} is close to 1, is true because at each stage d_k is not too small in comparison with r_{k+1}. In particular, $d_k r_{k+1} > \sqrt{r_{k+1}}$, which will be the dominant term in the estimate on m_{k+1}. Thus by only demanding that the matching occur on a small percentage of coordinates at each stage, the probability that this match occurs is large.

LEMMA 4.1: *Given $\epsilon > 0$, $\exists\, i$ such that $\forall k \geq i$, $m_{k+1} > 1-\epsilon/8$.*

Proof: Pick i such that

1. $r_k \geq r_i \geq i, \quad \forall k \geq i$,
2. $\frac{\epsilon^2}{16} r_{i+1} > 1$,
3. $e^{-(\sqrt{r_{i+1}})\epsilon^2/16} < \frac{1}{2}\ln 2$, and
4. $\sqrt{i} - \ln i \geq \frac{16}{\epsilon^2}\ln\left(\frac{2(1-\frac{\epsilon}{16})}{-\ln(1-\frac{\epsilon}{8})}\right)$.

Notice that for $0 < x < \frac{1}{2}\ln 2$, $e^{-2x} \le 1 - x \le e^{-x}$. Also, since $N_k \le (1-\epsilon/8)(1-\tilde{a}_{k+2})r_{k+1} + 1$, it follows that

$$(11)\qquad (1-\tilde{a}_{k+2})r_{k+1} - N_k \ \ge\ (\frac{\epsilon}{8} - \frac{\epsilon}{8}(1-\epsilon))r_{k+1} - 1$$

$$(12)\qquad \ge\ \frac{\epsilon^2}{8}r_{k+1} - 1$$

$$(13)\qquad \ge\ \frac{\epsilon^2}{16}r_{k+1}.$$

From these observations we have

$$(14)\qquad m_{k+1} \ \ge\ (1-(1-d_k)^{\frac{\epsilon^2}{16}r_{k+1}})^{(1-\frac{\epsilon}{16})r_{k+1}}$$

$$(15)\qquad \ge\ (1-e^{-\sqrt{r_{k+1}}\frac{\epsilon^2}{16}})^{(1-\frac{\epsilon}{16})r_{k+1}}$$

$$(16)\qquad \ge\ e^{-2(e^{-\sqrt{r_{k+1}}\frac{\epsilon^2}{16}})(1-\frac{\epsilon}{16})r_{k+1}}.$$

To show that $m_{k+1} \ge 1-\epsilon/8\ \forall k \ge i$ it suffices to choose i so that the following equation holds:

$$\sqrt{r_{i+1}} - \ln r_{i+1} \ge \frac{16}{\epsilon^2}\ln\left(\frac{2(1-\frac{\epsilon}{16})}{-\ln(1-\frac{\epsilon}{8})}\right).$$

But the function $\sqrt{x} - \ln x$ tends to infinity monotonically, so by condition 4 and the fact that $r_i \ge i$ we obtain this inequality.

The next lemma is needed to show that if $\sum \tilde{a}_k = \infty$, the total fraction of coordinates reserved from all the steps of the induction tends to 1. For a sequence $\{b_k\}$ set $f_k(x) = x + b_k(1-x)$ and $\tilde{f}_k(0) = f_1 \circ \cdots \circ f_k(0)$.

LEMMA 4.2: $\tilde{f}_k(0) = 1 - \prod_{i=1}^{k}(1-b_i)$.

Proof: The proof is by induction and is left to the reader.

Remark 4.1: If $\sum b_k = \infty$ and $0 < b_k \le b < 1$ then $\tilde{f}_k(0) \to 1$.

Proof of Theorem: Fix $\epsilon > 0$. Pick i such that $2^{-i} < \epsilon/16$, and such that the conclusion of the first lemma holds. Pick j such that, if $b_k = \tilde{a}_{j-k}$, then $\tilde{f}_j(0) > 1-\epsilon/8$. Let W_k be the total number of coordinates matched at the kth stage. In the first step of the algorithm, we reserve $\tilde{a}_j q_{j-1}$ coordinates of every G_{j-1} coset in every pair, and try to make these coordinates agree by rearranging G_{j-1} cosets around in the G_j word. After this step is done, we never move the reserved places again. Let x_k be the fraction of each G_j word that is reserved up through the kth step. $x_1 = \tilde{a}_j$. There are l^{2q_j} pairs, and $(1-\epsilon/8)$ of them fall in B_j (the set defined in the first lemma). At this point, nothing is reserved from the previous step, so

$$W_1 \geq l^{2q_j}\Big(1 - \frac{\epsilon}{8}\Big)r_j\Big(1 - \frac{\epsilon}{8}\Big)\tilde{a}_j q_{j-1}.$$

In the kth step of the algorithm, there are $l^{2q_j} r_{j-k+2} \cdots r_j x_{k-2}$ pairs of G_{j-k+1} cosets that have been reserved from steps 1 through $k-2$, and $\tilde{a}_{j-k+2}$ part of each G_{j-k+1} coset pair reserved from the previous step. Since X is iid, the occurrence of pairs from B_{j-k+1} as the name of an G_{j-k+1} coset will be at least $(1-\epsilon/8)(l^{2q_j})r_{j-k+2}\cdots r_j(1-x_{k-2})$ out of the total number of G_{j-k+1} coset pairs. If $(p,q) \in B_{j-k+1}$ then the number of coordinates matched is greater than

$$(1-\tilde{a}_{j-k+2})\tilde{a}_{j-k+1}(1-\epsilon/8)r_{j-k+1}q_{j-k}.$$

Notice $(1-\tilde{a}_{j-k+2})(1-x_{k-2}) = (1-x_{k-1})$, since $x_{k-1} = f_{j-k+2}(x_{k-2})$. Hence

$$W_k \geq (1-\epsilon/8)^2(l^{2q_j})q_j(1-x_{k-1})\tilde{a}_{j-k+1}.$$

But

$$\sum_{k=1}^{j-i} W_k \geq (1-\epsilon/8)^2(l^{2q_j})q_j\tilde{f}_{i+1,j}(0),$$

which gives a lower bound on the total number of coordinates that can be matched. Now

$$\int v_j^P(x,y)d\mu \times \mu \leq 1-(1-\epsilon/8)^3 + \epsilon/8 < \epsilon.$$

Note: There are two places where the estimates are affected if it is necessary to skip steps in the matching algorithm. First, in the technical lemma, if $\tilde{a}_{k+1} \leq 1/k^2$, we would calculate the probability of matching $[(1-\epsilon)r_{k+1}]+1$ of the G_k cosets instead of $[(1-\epsilon)(1-\tilde{a}_{k+2})r_{k+1}]+1$ of them, which can be done with only minor modifications in the argument. The other place that the argument is affected is when we count the number of coordinates matched. Suppose $\tilde{a}_{j-k+1}$ is not large enough. This time, in the kth step of the algorithm, we reserve

$$l^{2q_j} r_{j-k+1} \cdots r_j x_{k-1}$$

pairs of G_{j-k} cosets from steps 1 through $k-1$, since $x_{k-1} = x_{k-2}$. Notice that all of the coordinates reserved from the previous stages are whole G_{j-k} cosets. However, this gives the same final estimate on the number of coordinates matched.

COROLLARY 4.1 ([3]): *If r is entropy free, every r equivalence class contains a positive entropy action.*

Proof: Fix r entropy free. Given a G action T_0, there is a G action T, which is equivalent to T_0, and which has an r finitely determined action as a factor [8]. Let S_1 be the iid G action with measure $(\frac{1}{2}, \frac{1}{2})$. From the previous theorem, S is equivalent to S_1. Now construct a lift T_1 of S_1 that is equivalent to T. In other words, a lift that completes the following diagram:

$$\begin{array}{ccccc} T_0 & \xrightarrow{\sim} & T & \xrightarrow{\sim} & T_1 \\ & & \downarrow & & \downarrow \\ & & S & \xrightarrow{\sim} & S_1 \end{array} \tag{17}$$

Since S_1 is a factor of T_1, T_1 must have positive entropy, and T_1 is equivalent to T_0.

COROLLARY 4.2: *Any finite entropy Bernoulli process is standard when r is entropy free.*

Proof: Use the fact that for any iid process with equal weights on the symbols this is true, and the fact that factors of r finitely determined processes are r finitely determined [8].

Remark 4.2: It follows from this result and the fact that r equivalence is a restricted orbit equivalence that $h_r(T) = 0 \ \forall T$ when

$$\sum \frac{\log r_{k+1}}{q_k} = \infty.$$

Reference

[1] L. Dubins, J. Feldman, M. Smorodinsky and B. Tsirelson, *Decreasing sequences of σ-fields and a measure change for Brownian motion*, The Annals of Probability **24** (1996), 882–904.

[2] J. Feldman, *New K-automorphisms and a problem of Kakutani*, Israel Journal of Mathematics **24** (1976), 16–38.

[3] J. Feldman, Private communications, Annual Maryland Dynamics Conference, 1996.

[4] J. Feldman, *Standard orbit factors and entropy-free r equivalence*, Ergodic Theory and Dynamical Systems, submitted.

[5] A. Fieldsteel and J.R. Hasfura-Buenaga, *Dyadic equivalence to completely positive entropy*, Transactions of the American Mathematical Society, to appear.

[6] J. R. Hasfura-Buenaga, *Mixing for dyadic equivalence*, Acts Mathematica Universitatis Comenianae **64** (1970), 141–152.

[7] D. Heicklen, *Entropy and r equivalence*, Ergodic Theory and Dynamical Systems, to appear.

[8] J. Kammeyer and D. Rudolph, *Restricted orbit equivalence for discrete amenable group actions*, preprint.

[9] A. B. Katok, *Time change, monotone equivalence and standard dynamical systems*, Doklady Adakemii Nauk SSSR **223** (1975), 789–792.

[10] D. Ornstein, *Bernoulli shifts of the same entropy are isomorphic*, Advances in Mathematics **4** (1970), 337–352.

[11] D. Ornstein, D. Rudolph and B. Weiss, *Equivalence of measure preserving transformations*, Memoirs of the American Mathematical Society **37** (1982), 262.

[12] A. M. Stepin, *On an entropy invariant of decreasing sequences of measurable partitions*, Functional Analysis and its Applications **5** (1971), 80–84.

[13] A. M. Stepin, *Bernoulli shifts on groups and decreasing sequences of partitions*, Proceedings of the Third Japan-USSR Symposium on Probability Theory, Lecture Notes in Mathematics **550**, Springer-Verlag, Berlin, 1976, pp. 592–603.

[14] A. M. Vershik, *On lacunary isomorphism of monotone sequences of partitions*, Funkcionalnyi Analiiz i ego Prilozheniya **2** (1968), 17–21.

[15] A.M. Vershik, *Decreasing sequences of measurable partitions and their applications*, Doklady Adakemii Nauk SSSR **193** (1970), 748–751.

[16] A. M. Vershik, *A continual family of pairwise non-isomorphic dyadic sequences*, Functional Analysis and its Applications **5** (1971), 16–18.

[17] A. M. Vershik, *Four definitions of a scale of an automorphism*, Functional Analysis and its Applications **7** (1973), 1–17.

[18] A. M. Vershik, *Theory of decreasing sequences of measurable partitions*, St. Petersburg Mathematical Journal **6** (1995), 705–759.